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Remarks on the Wiener's compactification with applications to the classification theory

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Introduction.

Strictly speaking, the notion of the border of a Riemann surface can make sense either when we deal with a bordered Riemann surface, or when we consider the fuchsian group associated with a Riemann surface (, where the group must be of the second kind and the border corresponds to so-called free boundaries of the group). But a point on the border also can be characterized by the existence of a halfdisk-like neighbourhood on the bordered Riemann surface. Namely, a point on the border has a neighbourhood V such that $V \cap R$ is simply connected and $\partial(V \cap R)$ is a simple open curve, where R is the interior of the bordered surface.

Now utilizing this characterization, we may define the borderlike ideal boundary points on any compactification of a given Riemann surface. However we must then choose carefully the compactification which we will use. For example, the notion of the borderlike part on the chosen compactification should be a natural modification of that of the usual border, and some conditions we impose on the borderlike part should have close connections with certain properties of the fuchsian group associated with the surface.

Taking these things into account, we will use the Wiener's compactification.

Almost all proofs are omitted, and the details will appear in [8].

§ 1. Borderlike ideal boundary points.

First we recall the definition of the Wiener's compactification (cf. [2] and [7]). Let R be an open Riemann surface. Then we denote by W(R) the space of all real continuous bounded Wiener functions on R. If R belongs to the class O_G, then W(R) is coincident with the space of all real continuous bounded functions on R. And if not, we can decompose W(R) as follows;

$$W(R) = HB(R) + W_0(R),$$

where HB(R) is the space of all real bounded harmonic functions on R, and $W_0(R)$ is the space of all real continuous bounded Wiener potentials on R, or equivalently,

$$W_0(R) = \{ g\text{: real continuous bounded function on } R \text{ such that there is a potential } p \text{ satisfying the condition } |g| \leq p \text{ on } R \}.$$

Then there exists the unique compact Hausdorff space, say R^*_W, satisfying the following conditions;

1) R is dense open in R^*_W,
2) every f in W(R) can be continuously extended to R^*_W,
3) after such extensions, W(R) separates points in R^*_W.

We call this space R_W^* <u>the Wiener's compactification</u> of R.

Next if R does not belong to O_G, then set

$$\Gamma_W(R) = \{ p \in R_W^* - R : g(p) = 0 \text{ for every } g \in W_0(R) \}.$$

And if R belongs to O_G, then we assume that $\Gamma_W(R) = \emptyset$. This set $\Gamma_W(R)$ is called <u>the harmonic boundary</u> of R. Recall that the harmonic boundary is the support of the harmonic measure.

Now we call a point p in $\Gamma_W(R)$ a borderlike point, or simply a <u>b-point</u> of R if p has an open neighbourhood V in R_W^* satisfying the following conditions;

1) $V = \overline{(V \cap R)}^W - \overline{\partial(V \cap R)}^W$,

where and hereafter $\bar{X}^W$ means the closure of X in R_W^* and ∂X means the relative boundary of X in R,

2) $V \cap R$ is simply connected,

3) $\partial(V \cap R)$ is a simple (open) curve.

And set

$$d_W R = \{ p \in \Gamma_W(R) : p \text{ is a b-point of } R \}.$$

Then it is obvious that $d_W R$ is open in $\Gamma_W(R)$, and it can be seen that every point of $d_W R$ has vanishing harmonic measure.

Using this set $d_W R$, we can define the following three classes of Riemann surfaces.

$$SO_W^- = \{ R \notin O_G : d_W R = \Gamma_W(R) \}$$

$$SO_W^! = \{ R \notin O_G : d_W R \text{ is dense in } \Gamma_W(R) \}$$

$$O_W = \{R: d_w R = \emptyset\}.$$

Remarks. 1) R belongs to O_G if and only if $\Gamma_w(R) = \emptyset$, so the classes O_W and SO'_W are mutually disjoint.

2) By the definitions, it is clear that SO_W is contained in SO'_W.

3) O_{HB} is contained in O_W, for if R belongs to O_{HB}, then $\Gamma_w(R)$ consists of at most a single point of positive harmonic measure.

4) Because $d_w R$ is open, the harmonic measure of $d_w R$ equals to that of $\overline{d_w R}^w$, hence we can define the class SO'_W as follows;

$$SO'_W = \{R \notin O_G: \Gamma_w(R) - d_w R \text{ has vanishing harmonic measure}\}.$$

Example 1. SO_W is a proper subset of SO'_W. In fact, let U be the unit disc, $E = \{\exp[-\frac{1}{n} + \sqrt{-1}\frac{1}{k}]: n \in \mathbb{Z}^+, k \in \mathbb{Z}\}$ and $R = U - E$. Then we can see that R belongs to SO'_W, but not to SO_W.

Proposition 1. Let D be a subregion of a Riemann surface R such that ∂D consists of a countable number of disjoint simple curves not accumulating to any point of R. If D is of type SO_{HB}, then D belongs to SO'_W as a Riemann surface.

Of course, a subregion of type SO_{HB} does not necessarily belong to SO'_W as a Riemann surface without additional conditions on the relative boundary as in Proposition 1.

Now for the class SO_W, we note the following

<u>Proposition 2.</u> Let R be of finite genus g. Then R belongs to SO_W if and only if R can be considered as a subregion on a compact Riemann surface, say S, of the same genus g such that ∂R consists of

1) a finite set B of analytic simple closed curves, and

2) a relatively closed polar set E on the surface $\bar{R}$-B such that $\bar{E} \cap B$ is a finite set of points.

Here the closure is taken in S.

Roughly speaking, in case of finite genus, SO_W can be considered as the class of Riemann surfaces which are almost compact bordered.

<u>§ 2. On the type of fuchsian models.</u>

Hereafter we restrict ourselves on Riemann surfaces which have the hyperbolic universal covering surface. We may take the unit disc U as the universal covering surface, and denote by G = G(R) a fuchsian group associated with R on U. We call R is of type I and of type II, respectively, according as G is of the first kind and of the second kind (cf. [3] and [6]). Also, if the limit set L(G) of G has vanishing linear measure, we call R is of type II_0. It is well-known that if R is of type II, then R does not belong to O_G.

Now we can characterize the classes O_W and SO'_W by means of properties of fuchsian groups. First recall that if R does not belong to O_G, then the covering projection from U onto R can be extended to a continuous mapping, say P, from U^*_W onto R^*_W, and the identical automorphism of U can be extended to a continuous mapping, say I, from U_W^* onto $\bar{U}$, the usual closure of U in $\mathbb{C}$. Next let E = $\partial U - L(G)$, then E is empty or consists of a countable number of open arcs on ∂U, which we denote by $\{I_n\}_{n=1}^{\infty}$. Set $E' = \bigcup_{n=1}^{\infty} \overline{I_n}$, the union of the closures of I_n in $\mathbb{C}$. Then the crucial fact for our consideration is the following

Lemma 1. $E \subset I(P^{-1}(d_W R)) \subset E'$.

From this lemma, the following Proposition can be shown.

Proposition 3. R belongs to O_W if and only if R is of type I.

Here recall that we assume that R has the hyperbolic universal covering surface.

Also we can show the following

Theorem 1. SO_W' is coincident with the set of all Riemann surfaces of type II_0.

Corollary 1. The class O_W is quasiconformally invariant.

Corollary 2. Let D be as in Proposition 1. Then D is of type SO_{HB} if and only if D belongs to SO_W' as a Riemann surface.

Remark. It is well-known that the limit set of every non-elementary fuchsian group has positive capacity (cf. [5]).

Example 2. Using the Ahlfors-Beurling's celebrated example ([1]), we can show that the class SO_W' is not quasiconformally invariant. In fact, let f be a quasiconformal automorphism of U such that there is a compact set F on ∂U with zero linear measure whose image $f(F)$ has positive linear measure. And let F' be the union of F and a suitably chosen countable set on ∂U, and E be a countable set of points on U such that $\bar{E} \cap \partial U = F'$. Then $R = U-E$ and $R' = U-f(E)$ are quasiconformally equivalent, and we can conclude that R belongs to SO'_W, but R' does not.

Remarks. 1) The quasiconformal non-invariance of the class SO'_W implies that the limit set of certain fuchsian group with zero linear measure can be mapped by a quasiconformal automorphism of U on the limit set of a fuchsian group with positive linear measure. And it may be interesting to construct an explicit example of such a mapping.

2) It is still an open problem what happens on the structure of the Wiener's compactification, especially on the harmonic boundary, under quasiconformal mappings of Riemann surfaces.

§ 3. The classification of the double.

For a Riemann surface R of type II, we can consider the double of R, which we denote by $\hat{R}$. And set

$$DO_X = \{ R: R \text{ is of type II and } \hat{R} \in O_X \},$$

where X is G or HB or AB. Then first we have the following

Theorem 2. The following system of strict inclusion relations holds;

$$DO_{HB} \rightarrow SO'_W \rightarrow DO_{AB}.$$

In fact, we can show Theorem 2 by using Corollary 2 and Theorem 1.

Moreover we have the following

Proposition 4. Let R belong to SO'_W and consider R as a subregion of $\hat{R}$. And let $\{I_n\}_{n=1}^{\infty}$ be components of ∂R, then we have

$$\Gamma_w(\hat{R}) \subset \overline{\partial R}^w - \bigcup_{n=1}^{\infty} \overline{I_n}^w .$$

Also note the following

Lemma 2. If R belongs to SO_W, then the number of components of ∂R is finite in number.

Now by using Proposition 3 and Lemma 2, we can easily show the following

Theorem 3. SO_W is a proper subset of DO_G.

And using Theorem 3, we can show the following

Theorem 4. The class SO_W is quasiconformally invariant.

(Outline of the proof) Let R belong to SO_W and another R' be quasiconformal equivalent to R. Then we can see that $\hat{R}'$ belongs to O_G. Hence using Lemma 2 we can conclude the assertion.

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